Convex Optimization for Low-Complexity Spatial Antenna Arrays

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Abstract---Conventional smart antennas create flexible beam patterns using weights that have both magnitude and phase. This requires expensive hardware in the form of individual receivers and transmitters, high-speed D/A and A/D converters and capable DSP or FPGA processors. This paper looks at the use of low-complexity spatial antenna arrays that can create reasonably complex antenna patterns using a phased array approach. Convex optimization is applied to solve the highly non-linear optimization problem. Square array geometries were studied in depth by applying various penalty functions. We found that convex optimization is a novel and effective way to compute the complex antenna weights, and that this low-complexity approach is an interesting alternative to more expensive smart antennas

Keywords-convex optimization; spatial antenna; penalty functions; phased array; antenna weights

I. INTRODUCTION

We are investigating a hybrid analog/digital beam former (ADBF). Traditional analog beam formers (ABF) steer a single beam using a single transceiver, power splitter/combiner, and electronically controlled analog phase shifters. In contrast modern digital beam formed arrays (DBF) or 'smart antennas' use separate transceiver chains, A/D and D/A converters, and DSPs [1]. This enables direct digital control of array weights to optimize criteria such as MSE. DBF technologies are effective but are not suitable for every application due to their relative high cost and complexity. The technique we are addressing is similar to ADBF approaches using a single transceiver and analog phase shifters, but phase shifters are digitally controlled to provide flexible multiple nulls and beams defined in real time.

The synthesis and design of antenna arrays has been extensively studied over many years. Many fast and efficient methods for finding the antenna weights have been developed [1]-[6]. However, convex optimization techniques have rarely been used for these types of problems [2]. This paper illustrates its application to antenna array development and shows that it has some important advantages including ease of use and the ability to apply different penalty functions for solving the least squares problem. Although much previous research deals exclusively with Uniform Linear Arrays (ULAs), the convex optimization methods facilitate array solutions for spatial arrays as well. Geometries such as square, circular, or 'Y' arrays provide many advantages including maximum output power and high Signal-to-Noise Ratio (SNR) [1] and [3]. This paper focuses on solutions for square arrays, but results can be easily extended to other geometries including the ULA.

Complex weights can be implemented by cascading digitally controlled variable gain amplifiers and phase shifters. (In this paper we refer to this as the 'unconstrained' case). However, bidirectional amplifiers add cost and complexity and should be avoided if possible. A simple alternative is to use stepping attenuators, which are bidirectional but constrain weight magnitude to be unity or less (we refer to this as the 'constrained' case). This approach shown in Fig. 1. A possible disadvantage is that the attenuators may provide unacceptable losses in some systems that have tight power budgets. A third approach is to completely eliminate the attenuators and develop antenna patterns using only phase shifters. However, this is not a convex problem [2] and is not addressed in this paper.

II. CONVEX OPTIMIZATION PROBLEM AND CVX TOOLBOX

A set is convex if for any pair of its points, the line joining these two points lies in the set. A function f is convex on a convex domain if $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$ for

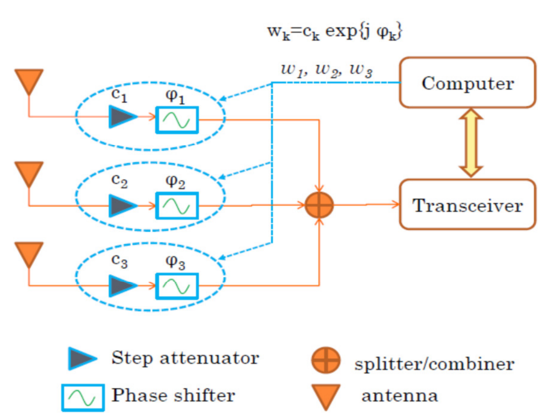

Figure 1. Hybrid system using attenuators and phase shifters

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$\lambda: 0 \le \lambda \le 1$. The convex optimization problem involves minimizing a convex function over its domain, which is always a convex set. The biggest advantage with convex functions is that the local minimum is always the global minimum. Many functions that we come across are convex functions. This includes functions such as affine functions, $a^T x + b$ where a and x are vectors and b is a scalar, as well as quadratic functions like $x^T R x$ (given that R is a positive semi definite matrix). Norms of vectors like $||x||$ (which includes all norms like 1-norm, Euclidian Norm, and infinity norm). Many combinations of all these functions can also be convex and this can be determined easily from convex function techniques as described in [7].The solutions developed in this paper were computed using CVX, which is a Matlab-based modeling system for convex optimization that allows constraints and objectives to be specified using standard Matlab expression syntax [8], [15]. In order to use CVX to solve optimization problems one must follow disciplined convex programming rules, the details of which can be found in [9]. Generally, the notation

$$\begin{aligned} &minimize\ f_0(x) \\ &subject\ to\ f_i(x) \le 0, \quad i = 1,2,\dots,m \\ &\qquad h_i(x) = 0, \quad i = 1,2\dots,p \end{aligned} \tag{1}$$

is used to describe the problem of finding the minimizer for the function f_0 subject to the m inequality and p equality constraints. This minimizing problem is called a convex optimization problem only if $f_0, f_1, f_2, \dots, f_m$are all convex functions and h_i are all affine functions.

III. Penalty Functions in Convex Optimization

Penalty functions are simply cost functions that penalize errors. The use of penalty functions in antenna array design is a general idea and though there are many penalty functions available for use, the “2-norm” function is frequently used in antenna array design. This section introduces two important penalty functions other than the 2-norm and in the later sections the performance of an array based on these penalty functions is discussed.

The simplest norm approximation problem is an unconstrained problem of the form

$$minimize\ \ ||Ax - b|| \tag{2}$$

where $A \in \boldsymbol{R}^{mxn}$ (m are the number of equations and n are the number of variables)and $b \in \boldsymbol{R}^m$ are the data available from the problem and $x \in R^n$ is the variable. The solution obtained is sometimes called an approximate solution of $Ax \approx b$. The vector $r = Ax - b$ is called the residual for the problem and the smaller the residual value is then the better the solution for the approximation problem.

There is always at least one optimal solution for the norm approximation problem. The optimal residual value is zero iff $b \in \Re(A)$, where $\Re(A)$ is the range space of A. The problem becomes interesting when $b \notin \Re(A)$. When $m = n$, the problem is said to be completely determined and the optimal point is simply $A^{-1}b$ if the matrix A is invertible. For a system of linear equations (that is, a matrix equation $Ax = b$), the system is underdetermined if there is an infinite number of solutions. If A is mxn with $m < n$, either there are free variables and an infinite numbers of solutions (underdetermined), or the system is inconsistent and there are no solutions. To solve this problem various penalty functions can be used.

The penalty function approximation problem has the form

$$\begin{aligned} &minimize\ \ \Phi(r_1) + \dots + \Phi(r_m) \\ &subject\ to\ \ r = Ax - b \end{aligned} \tag{3}$$

where, Φ: $\boldsymbol{R} \rightarrow \boldsymbol{R}$ is called the (residual) penalty function. When Φ is convex, the above problem is a convex optimization problem. There are many convex penalty functions one of which is the 1-norm penalty function:

$$\Phi(u) = |u| \tag{4}$$

As described in [10], 1-norm penalty function puts relatively larger emphasis on small residuals as compared to the 2-norm function. The optimal residual value found by 1-norm functions will tend to have more zero residuals or very small residuals. It therefore gives the sparse solution for the residuals and thus, in the context of estimation, is called a *robust estimator*.

There is always the problem of outliers in any regression or estimation problem and when outliers occur, any estimate of x will be associated with a residual vector with some large components. We would like to actually remove the outliers to make the data more flawless and this can be done by choosing some threshold functions. The Huber function is one such penalty function:

$$\Phi_{hub}(u) = \begin{cases} u^2 & ,|u| \le K \\ K(2|u| - K) & ,|u| > K \end{cases} \tag{5}$$

As indicated in (5), this function behaves like a quadratic penalty function for residuals lower than a fixed value, K, and it behaves like a 1-norm penalty function for residuals greater than the fixed value. This paper will demonstrate the use of these two important penalty functions for solving antenna array design problems.

IV. Optimization Problem

This paper considers the array to comprise N=4 antennas. However, the solution approach is valid for larger arrays. If the signal is arriving from direction θ, then the output is: $y(\theta) = w^{H} \cdot a(\theta)$. Here, $a(\theta)$ is the steering vector, which, for a square array is:

$$[exp\{j(kd\ cos\theta)\}, exp\{-j(kd\ cos\theta)\}, exp\{j(kd\ sin\theta), exp\{-j(kd\ sin\theta)\}]^{T} \quad (6)$$

The complex weight vector w^H is given by

$$w^{H} = [c_1 exp(j\phi_1), c_2 exp(j\phi_2), c_3 exp(j\phi_3), c_4 exp(j\phi_4)] \quad (7)$$

where, k=2π/λ is the wave number and d is the inter-element spacing. If we want to control the output in more than M directions then the outputs are

$$y = [y(\theta_1), y(\theta_2), \dots, y(\theta_2)] = w^{H} A + v \quad (8)$$

where A is an NxM matrix of steering vectors and v is white noise,

$$A = [a(\theta_1), a(\theta_2), \dots, a(\theta_M)] \quad (9)$$

To find the optimal weights for minimum MSE, the error is: $\xi^{H} = w^{H} A - u$, where u is an 1xM vector of desired array responses and the squared error $\xi^{H}\xi$, is the objective function to be minimized and ^{H} is Hermetian transpose.

The minimum MSE solution is simply: $w = uA^{\dagger}$, where $A^{\dagger}$ is the pseudo-inverse. For the constrained case in Fig. 1, we seek to minimize $\xi^{H}\xi$ under the constraint that all $c_k \leq 1$. This difficult non-linear problem can be solved using convex optimization as a 2-norm function is a convex function. The completely determined unconstrained problem can be solved using 2-norm penalty function or using the Matlab's *mrdivide* operator: '\'. The interesting use of penalty functions can be understood when the problem is underdetermined and then the unconstrained problem can be solved in CVX using the penalty functions discussed earlier or the Matlab's *mrdivide* operator (which now finds the MSE solution by finding the pseudo inverse). We compare the results of all these methods in solving for the design parameters in an antenna array. The constrained problem is implemented in CVX as: $abs(w) <= 1$, which is a convex set.

V. Convex Solution To the Optimization Problem

A. Underdetermined Problem

Conventional adaptive arrays have no significant constraints on the magnitude or phase of the weights and so have greater control over beam forming. The optimized unconstrained array results are used for comparison with the constrained-weight arrays being studied here. With $N = 4$ elements, many geometries can be formed but as described in [1], a square array gives us good performance and is used to obtain all the results discussed in this paper. The degrees of freedom of an N-antenna array allow us to control the direction of n main beams and m nulls where $n + m <= N$. We will first try to see the performance of the array with one main beam in a desired direction and two nulls in the interference directions**.** Please note that this is an underdetermined problem. The unconstrained problem for a least squares problem is easy to solve using a pseudo-inverse. Since this is an underdetermined problem, we cannot use the 2-norm penalty function to solve for the unconstrained problem given by:

$$minimize\ \left(norm(\xi)\right) \quad (10)$$

Using 2-norm in CVX for this problem results in an error. The importance of penalty functions can be observed in solving this problem and as described earlier in this paper, 1-norm and Huber penalty functions can be used to solve for underdetermined problems. It is always easy to reduce the original problem of finding a solution to the underdetermined problem $Ax = b$ to a sub problem $\tilde{A}\tilde{x} = b$, where $\tilde{A}$ is the m x m sub matrix of A which can be obtained by selecting only those columns of A which correspond to the m indices (out of $1, \dots, n$) which are to be non zero components of x and $\tilde{x}$ is the sub vector of x containing the m selected components. As described in [10], if $\tilde{A}$ is nonsingular, then we can get the solution simply by inverting $\tilde{A}$: $\tilde{x} = \tilde{A}^{-1}b$. If $\tilde{A}$ is singular and b is not in the range space of A, then there is no feasible x with the chosen set of non zero components. If $b \in \Re(\tilde{A})$, and $\tilde{A}$ is singular, then a feasible solution with fewer than m nonzero components exists. With simple knowledge of combinations, one can clearly understand the point drawn out of this in [10] that, this approach requires examining and comparing all $n!/(m!\,(n-m)!)$ choices of m nonzero coefficients out of n coefficients in x. Using 1-norm penalty function avoids all these complications and gives us a good heuristic for finding a sparse solution to the unconstrained underdetermined problem.

We assume that we have a-priori knowledge about directions to signal and interference sources are and also the direction of main sources. We attempt to find the optimal weights that give us a response close to the desired response. Similarly, the Huber penalty function can be used to frame a convex optimization problem for unconstrained case. An inter-element spacing of $\lambda/2$ is used as suggested by many authors for solving unconstrained problem [2], [3], [6], [11], and [12] .We sweep the whole space with the main beam from 0-360^0 to evaluate the complete response of the antenna array Recall that we have a demand on our array to produce only two nulls. This paper studies a particular underdetermined problem where the demand is for a single main beam and two nulls at the interference directions. Radiation patterns in one of the cases, where a main beam must be formed at 27.73^0 and nulls at 63.73^0 and 99.73^0, for all the penalty functions discussed are given in Fig. 3. Ideally, one would like a single main beam in the direction of the source and nulls everywhere else. Fig. 3 indicates that there are directions where the array provides gain to those directions where there are no sources or interferences. This, at first, may seem a trivial issue but in applications where significant multipath or white noise is present, the side lobes are detrimental to the system.

We tested the array for an inter-element spacing of $\lambda/4$, which has been suggested in [13] to decrease the effect of quasi-grating lobe, and the radiation plots for this spacing are given in Fig. 4. As can be clearly seen from the plots, the main lobes have been formed in the direction of source and nulls are formed in the direction of interferences using all methods.

A useful measure to compare all the methods under discussion is SNR where the gain in the direction of main beam is used to calculate the signal power and the gain in all other directions is used for calculating the noise power. The mean SNR is calculated by averaging results from rotating the beams through 360 degrees. Results for both an inter-element spacing of $\lambda/2$ and $\lambda/4$ are given in Table I and Table II. Large negative SNR values are due to the unconventional definition, but are nevertheless useful for

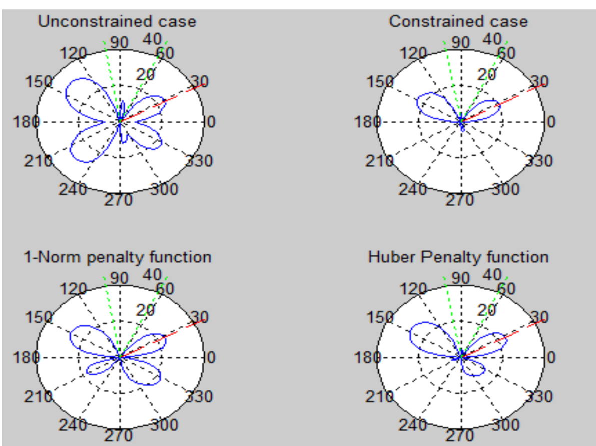

Figure 3. Radiation pattern for the example with an inter-element spacing of λ/2. (Red line is direction of main beam and green is the direction of nulls)

TABLE I. TABLE OF MEAN SNR VALUES FOR D= λ/2 FOR A PARTICULAR UNDERDETERMINED PROBLEM

Method	Mean SNR (in dB)
Constrained	-6.39
Unconstrained case using Matlab	-9.83
Unconstrained case using 1-norm penalty function	-8.05
Unconstrained case using Huber penalty function	-7.89

The SNR values are negative because of the problem set up.

spacing of $\lambda/2$ are better than in the case of an inter-element spacing of $\lambda/4$ and so if the user is applying this in a communication system, where multipath propagation must be taken care of, then we propose that an inter-element spacing of $\lambda/2$ is better.

The SNR values also emphasize the importance of the constrained case, which is the modified system discussed in this paper. The values for the constrained case are better than the unconstrained case in terms of SNR and also the difference in its performance is less than 3dB for the different inter-element spacing where as there is a difference of greater than 9dB for all other unconstrained methods.

B. *Completely Determined Problem*

A completely determined problem has also been studied where all the degrees of freedom of an antenna array have been completely utilized for its performance. Now, it is assumed that three interference sources are present and one desired source is present, which means three nulls in three different directions and one main beam in a desired direction. The problem has been studied in the same way as the underdetermined problem. Since it has already been determined that an inter element spacing of $\lambda/2$ is better for a communications problem, this problem has been studied with

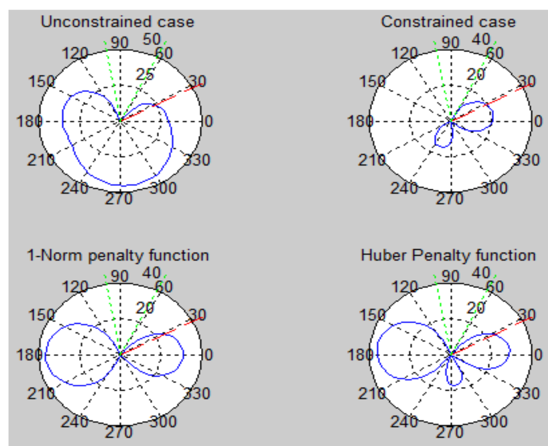

Figure 4. Radiation pattern for the example with an inter-element spacing of λ/4. (Red line gives the direction of source and green give the direction of nulls)

TABLE II. TABLE OF MEAN SNR VALUES FOR D= λ/4 FOR A PARTICULAR UNDERDETERMINED PROBLEM

Method	Mean SNR (in dB)
Constrained	-7.60
Unconstrained case using Matlab	-16.21
Unconstrained case using 1-norm penalty function	-12.65
Unconstrained case using Huber penalty function	-12.45

this inter-element spacing. The same case as described in the underdetermined problem is used for the plots but now an additional null is requested at 135.73^0. The radiation patterns for all the cases under consideration are shown in Fig. 5.

The SNR values are given in Table III. In the case of unconstrained adaptive weight vectors, theory suggests that an n element array can completely null $n-1$ point interference sources [14]. The SNR values for the unconstrained cases are now worse than it was for the underdetermined problem and this is due to the fundamental theorem discussed earlier. In the underdetermined problem, the unused degree of freedom was helping the array to lower the noise power. In the completely determined problem all the degrees of freedom are being used for beam forming and so the SNR values have decreased by more than 10dB in all unconstrained cases. The constrained case still has the same mean SNR as it had with the underdetermined problem and this again underlines the importance of the modified system in terms of consistency in addition to low complexity. The SNR results as well as the radiation plots show that in case of completely determined problem, all the unconstrained cases perform similarly.

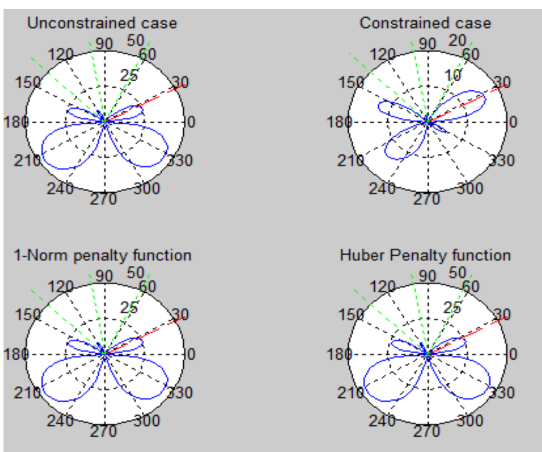

Figure 5. Radiation pattern plots for completely determined problem with an inter-element spacing of λ/2

TABLE III. TABLE OF MEAN SNR VALUES FOR D= λ /2 FOR A COMPLETELY DETERMINED PROBLEM

Method	Mean SNR (in dB)
Constrained	-6.40
Unconstrained case using Matlab	-14.98
Unconstrained case using 1-norm penalty function	-14.98
Unconstrained case using Huber penalty function	-14.98

This suggests the use of 1-norm and Huber penalty functions in the synthesis of antenna array performance as they led to results that were either better (in the underdetermined case) or equal (in the completely determined case) in performance, in the terms of SNR, as compared to a general penalty function like the 2-norm function.

VI. CONCLUSION

Convex optimization is shown to be a useful tool in antenna array design and can be used effectively for deciding many design parameters such as inter-element spacing. Penalty functions like Huber penalty function and the 1-norm penalty function are useful for improving side lobe performance, and they give better results than the commonly used 2-norm function in terms of SNR in the underdetermined case. Finally, the convex optimization approach offers an interesting alternative for designing not only ULA's but also arbitrary spatial arrays.

REFERENCES

[1] Srinivasa Yasasvy Sateesh Bhamidipati, and Robert F. Kubichek, "Low complexity hybrid analog/digital phased array for ad-hoc networks", in press.
[2] Herve Lebret, and Stephen Boyd, "Antenna array pattern synthesis via convex optimization", IEEE transactions on signal processing, vol. 45, no. 3, March 1997.
[3] Peter Joseph Bevelacqua, Antenna Arrays: Performance Limits And Geometry Optimization, (Doctoral Dissertation), Arizona State University; 2008. 171p.
[4] O.M. Bucci, D. D'Elia, G. Mazzarella, and G. Panatiello, "Antenna pattern synthesis: A new general approach," Proc. IEEE, vol. 82,pp. 358-371, March 1994.
[5] S. A. Schelkunoff, "A mathematical theory of linear arrays," Bell Syst. Tech. J.,vol. 22, no. 1,pp. 80-107, Jan. 1943
[6] M. Mouhamadou, P. Vaudon, and M. Rammal, "Smart antenna array patterns synthesis: Null steering and multi-user beamforming by phase control," Progress in electromagnetic research, PIER 60, pp. 95-106, 2006.
[7] Stephen Boyd, and Lieven Vandenberghe, Convex Optimization, Cambridge University Press, 2009, pp. 127-152.
[8] *CVX Research*, Michael C. Grant, Retrieved February 16th, 2012, from http://cvxr.com/cvx/
[9] *CVX Research*, Michael C. Grant, Retrieved February 16th, 2012, from http://cvxr.com/cvx/cvx_usrguide.pdf
[10] Stephen Boyd, and Lieven Vandenberghe, Convex Optimization, Cambridge University Press, 2009, pp. 294-305.
[11] Frank Gross, Smart Antennas for Wireless Communications with MATLAB, McGraw Hill, pp. 65-101, 2005.
[12] Constantine A. Balanis and Panayiotis I. Ioannides, Introduction to Smart Antennas, 1st ed., Morgan & Claypool, pp. 107-124, 2007.
[13] T. Bao Vu, Smart Antenna System Architecture and Hardware Implementation, CRC press LLC, 2002, ch. 22.
[14] Steven T. Smith, "Optimum phase-only adaptive nulling," IEEE transactions on signal processing, vol. 47, no. 7, July 1999.
[15] M. Grant and S. Boyd. CVX: Matlab software for disciplined convex programming, version 1.21. ../../cvx, April 2011

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