Abstract— In this paper, I briefly review two problems from cosmology where computational mathematics could play a decisive role. The first problem concerns "Time's Arrow", i.e. the question why time seems to have a definite direction in spite of the fact that the underlying laws of nature appear to be essentially time-symmetric. The second problem concerns the origin of so called "dark energy": it may be that the field equations of general relativity have a much more complicated structure than they are usually thought to have. In both cases, computational mathematics could contribute, not just by solving given problems, but also by formulating the models properly.

Index Terms— cosmology, dark energy, entropy, graph, time’s arrow

I. INTRODUCTION

In recent decades, the methods and impact of computational mathematics have, as an inevitable consequence of the rapid development of computer technology, spread to many new fields of research. In this paper, I will argue that cosmology is a field where there is still a lot to be done.

Cosmology in its modern form can be said to begin very soon after the birth of general relativity approximately one hundred years ago. However, general relativity itself developed rather slowly for many years, mainly because the machinery of classical differential geometry is so heavy that it was simply almost impossible to work out solutions except for a very limited number of situations. Again, this has now changed a lot, starting from the seventies.

In this note I will briefly discuss two examples. The first is the problem of Time’s Arrow, i.e. the question why time seems to be directed, while the underlying microscopic laws of nature seem to be essentially time symmetric. The second problem concerns the origin of dark energy or, more specifically, the fact that our universe seems to expand at an accelerating pace which can not be explained by traditional general relativity.

What I suggest is that a slight shift in our perspective together with computational power can open up completely new horizons. In particular, in both cases computational mathematics will be needed, not just to solve given problems, but also to design the correct models.

This paper is written mainly for non-cosmologists and it is my explicit ambition to avoid technicalities. The models are simplified and I only hint at possible solutions (some references are included). Rather, my ambition is to attract attention to what could be a very fruitful field of research for computational mathematics.

II. COSMOLOGICAL MODELS

Let us start from one of the simplest possible models for cosmology, namely the closed Friedmann model (see [2], [5]). In this model, the universe starts from the "Big Bang" in a state of zero volume, then expands up to a maximal size and after that shrinks down to a "Big Crunch" in a symmetric way. Thus, space-time forms a compact, isotropic, pseudo-Riemannian 4-manifold where the space-part at every moment of time can be thought of as a 3-sphere with radius $x(t)$. The metric is given by

$$ds^2 = -dt^2 + x(t)^2 (d\chi^2 + \sin^2 \chi(d\theta^2 + \sin^2 \theta d\phi))$$

in terms of spherical coordinates, where $x(t)$ takes the form of a cycloid (see figure 1). Note the somewhat unconventional choice of time-interval $[-T_0, T_0]$ in the figure, which emphasizes the time-symmetry of the model.

Remark 1 The closed Friedmann model is nowadays not the most popular one. In fact, it has been clear for a long time that it has to be modified. In particular, ever since the discovery of the accelerating expansion of our universe, many authors have argued that space-time can not be closed (compact) at all, but must continue to expand for ever.

In this paper, I will stick to closed models for two reasons. The first one is simplicity: a closed model is essentially finite or at least compact. The second reason is deeper: in Section 5 it will become clear that compactness of space-time is not just a simplification, but in fact an active ingredient in the kind of explanation that I suggest for the accelerating expansion.

III. THE MULTIVERSE

In recent years, the idea of the "Multiverse" has become increasingly popular. In fact, the concept of "parallel worlds" is now so widespread that it is sometimes difficult to know if...
it refers to science or fiction. In this paper I will only make use of this idea in the following very restricted sense:

The Idea of the Multiverse: From the supposedly unique state at the Big Bang to the likewise unique state at the Big Crunch, an enormous number of different developments are possible, and to each such development, quantum mechanics attribute a certain probability. The totality of all such developments is called the Multiverse. Given a certain state at a certain time, the future development of that state is not unique, but many different futures are possible. It is inherent in the idea of the Multiverse that all these developments are considered to be equally real, and that what we perceive as our world at this moment is just one of all these possibilities.

It should be noted that the meaning of the word “state” in the multiverse context is a bit complicated. In fact, the states here can neither be considered to be classical nor quantum mechanical states, but rather represent a kind of semi-classical approximation.

The idea of the Multiverse is intimately connected with time’s arrow. In fact, if different futures are possible, and if the laws of physics are essentially time symmetric, why do we then perceive the past as unique? The answer that I suggest is that the laws of physics actually do allow for different pasts as well as different futures, and that the fact that this is never observed in real life is a consequence of properties of the concept of entropy, combined with the structure of the Multiverse as a probability space.

In view of the enormous complexity of the Multiverse, it is inevitable to use drastic simplifications. A particularly simple model for the Multiverse is to consider it to be a huge graph. First of all we suppose that time is discrete and integer valued. We consider a finite number \( N \) of particles moving in time from \(-T_0\) to \(T_0\), and at each moment of time, there is a finite number of possible states for these particles. The set of all such states constitute the vertices of the graph. Instead of trying to assign to each possible development a certain probability we just assume that the transition from a certain state at time \( t \) to another state at time \( t + 1 \) is either possible or impossible. The edges of the graph then simply connect states at adjacent moments of time which are accessible from one another. Given this set-up, we can define a universe to be a path in this graph which connects the unique states at times \(-T_0\) and \(T_0\), such that time is monotonic along the path.

Actually, this model has to be slightly modified close to the end-points: when the multiverse is very small, quantum effects dominate and almost everything could happen. On the other hand, it would not be realistic to assume that everything could happen with the same probability, so something extra is needed (see the next section).

IV. Time’s Arrow

Ever since Boltzmann ([1]) it has been clear that time asymmetry has something to do with the second law of thermodynamics, i.e. the statement that the entropy of our universe is increasing with time. Although it is not so easy to explain this relation exactly, I will for the purpose of this note, identify the problem of Time’s Arrow with the problem of explaining the behavior of the entropy. In a famous paper from 1962, Gold suggested that the growth of entropy is caused by the expansion of the universe ([3]). If this were the case, time’s arrow would switch direction if the universe would start to contract, and the behavior of time would be symmetric. However, nowadays few cosmologists believe that this is what is going to happen, so the efforts since that time have mainly been spent on trying to explain why the behavior is not symmetric.

There has been a very large number of different attempts to explain time asymmetry (see [4], [9]). But up to this day, it has not even been possible to reach an agreement about where to look for the solution: is it a quantum mechanical or an essentially classical problem? Is it a problem about the boundary conditions of the universe or is it something hidden within the dynamical laws? According to the perhaps leading authority in this area, H. Dieter Zeh, ([9]), the riddle of time has turned out to be remarkably illusive.

In figure 2, we can see three possible scenarios for the entropy. To explain time asymmetry within the kind of multiverse discussed in the previous section, amounts to proving that behavior of type 2 and 3 is enormously much more probable than behavior of type 1 (Gold’s type). Note that from this point of view, the multiverse is still completely time symmetric, with an equal number universes with increasing entropy (Time’s Arrow points to the right) as with decreasing entropy (Time’s Arrow points to the left). However, for an observer confined to one universe the behavior would still appear to be asymmetric.

To carry this plan out in the present setting however, we must first somehow relate the supposedly time symmetric dynamics of the multiverse to the entropy in the simplified combinatorial multiverse of the previous section.

The most important property of entropy is summarized in Boltzmann’s famous formula:
\[ S = k_B \log \Omega, \tag{4.1} \]

where \( \Omega \) denotes the number of states with entropy \( S \), and \( k_B \) is Boltzmann’s constant. Inverting this formula, we can also write

\[ \Omega = W^S \quad \text{where} \quad W = e^{\frac{1}{k_B}S}. \tag{4.2} \]

Thus, the number of possible states is an exponentially growing function of the entropy \( S \). If the number of states is finite, this can clearly hold only as long as the entropy is far from being maximal. Thus, it will tacitly be assumed that our multiverse, during the whole time interval from \(-T_0\) to \( T_0\), is in a comparatively ordered state so that (4.2) applies.

\( W \) will, for a realistic multiverse, be very large, and it is a complicated question how \( W \) should depend on time. But in any case, given any state (with entropy far from being maximal), the set of close-by states which can be accessed from the given one (in both directions of time) will be dominated by states with higher entropy, simply because there are so many more of them. This is the intuitive reason behind the following very simplified

**Dynamical Assumption:** Given any state \( \Xi \) of a universe with entropy \( S \) at time \( t \) (not too close to the end-points), every accessible state at times \( t \pm \Delta t \) will have entropy \( S \pm \Delta S \). The number \( K \) of accessible states with higher entropy is enormously much larger than the number of accessible states with lower entropy (in both directions of time).

The assumption that the entropy can only change by \( \pm \Delta S \) and hence is integer-value is of course again an extreme simplification. But for the purpose of explaining the origin of Time’s Arrow it seems to be a reasonable one.

Technically speaking, the problem of Time’s Arrow can now be viewed as the graph theoretical problem of computing the number paths with different behaviors of the entropy (like in figure 2). However, we must still add some assumption about what happens near the end-points \(-T_0\) and \( T_0\). A very natural such assumption is to say that during the very first (and last) moments of time, a universe can pass from the end-state with zero entropy to any state \( \Xi \), but that the probability weight for such a behavior is rapidly decreasing with the entropy of \( \Xi \). The exact nature of this probability distribution is not important, but a rather natural choice would be to assume that the probability for the entropy of \( \Xi \) should be Poisson-distributed.

This is essentially the set-up for explaining time-asymmetry that I suggest. It would lead too far to go into the technical arguments here so I simply refer to [7] (see also [8] for a very small numerical example). However, it should be emphasized that the arguments in both these references are based on still a few additional simplifying heuristic assumptions, e.g. only universes where Time’s Arrow switches direction once are considered and only for certain choices of the parameters \( N \), \( T_0 \) and \( W \). A convincing argument which takes into account all possible paths from \(-T_0\) to \( T_0\) is still missing. So the problem I suggest to computational mathematics is to construct a reasonably large model where it can actually be proved, mathematically or numerically, that highly asymmetric paths (similar to paths of type 2 & 3 in figure 2) dominate among all possible universes.

**V. The Accelerating Expansion of the Universe**

Let us now return to the simple closed Friedmann model in Section 2. The cycloidal behavior in figure 1 is obtained by solving Einstein’s field equation with a certain mass-density. However, with the discovery of the accelerating expansion of our universe, this model is no longer adequate to explain observations. In fact, accelerating expansion manifests itself by making the function \( x(t) \) in (2.1) convex, in contrast with the obviously concave cycloid in figure 1.

Einstein himself realized (although in a slightly different context) that the only way in which his field equations could be modified without getting into immediate contradiction with observations was to add an extra term, containing the so called cosmological constant (see [5]). He did so very reluctantly, since this term spoils much of the naturalness of his theory. And when he was given the opportunity, he was happy to return to the original equations. Nowadays however, with the discovery of the accelerating expansion, this extra term seems to be something that we simply will have to accept. So the question is rather how we can adapt to it.

The most common way of explaining the cosmological constant (which by the way does not really have to be a constant but could be a slowly varying entity) is to postulate some previously unknown kind of energy, usually referred to as "dark energy" (see for instance [6]). But so far we do not know what this strange kind of energy could be. So perhaps we should look for the solution within the field equations themselves?

Let us start by investigating the field equations in vacuum which are simply given by

\[ \text{Ricci} = R_{ij} = 0 \tag{5.1} \]

i.e. by the vanishing of the Ricci tensor. It is well known that these equations, as well as the more general version in the presence of matter, can be deduced from the Hilbert-Palatino variational principle. Thus, solutions of the field equations are simply the stationary metrics for

\[ \delta \int R dV = 0, \tag{5.2} \]

where \( R \) is the scalar curvature. The Hilbert-Palatino principle is a very elegant and sometimes powerful tool in general relativity. But it does not really give much insight into the underlying physics. Also, it is definitely not the only variational principle which can reproduce the field equations in vacuum.

Examples of such variational principles are given by minimizing

\[ R^2 + \mu |\text{Ricci}|^2 \tag{5.3} \]

where \( \mu \) is some positive constant and the norm is the tensor norm.
\[ \|\text{Ricci}\| = R_j^i R_j^i. \] (5.4)

Taken from the usual point of view, this kind of variational principle has many drawbacks as compared to the Hilbert-Palatino principle: not only is it considerably less suited for computations, but also, from the point of view of standard general relativity, it may be considered to be just a reformulation of the vanishing of the Ricci tensor.

However, this perspective changes if we instead consider a universe with a priori fixed volume. This idea may be somewhat alien to the traditional idea of space-time as an empty arena where particles perform their actions. But it is certainly not strange to modern physics, where space-time itself is an active constituent.

If the total four-volume is fixed, it may not be possible to find a metric with vanishing Ricci-tensor, and the problem of minimizing (5.3) becomes a non-trivial one, even in the case of an empty universe (containing no matter). In addition, this perspective offers a way of restoring the simplicity of Einstein’s original theory in the sense that the cosmological constant is no longer an arbitrary entity, but is determined by the total volume (and in the general case, by the distribution of mass in the universe).

VI. THE EULER-LAGRANGE EQUATION

Let us now return to the metric in (2.1) and try to determine the function \( x(t) \) so as to minimize the functional in (5.3) subject to the condition

\[ \int dV \propto \int_{-T_0}^{T_0} x(t)^3 \, dt = V_0. \] (6.1)

The traditional way to attack this problem is to look at the Euler-Lagrange equation, i.e. the stationary solutions for

\[ \delta \left( \int (R^2 + \mu \|\text{Ricci}\|) dV + \lambda \int dV \right) = 0. \] (6.2)

This leads to an ordinary differential equation for the function \( x(t) \). To derive this equation by hand turns out to be a lot of work. Using Mathematica however, we arrive at the following forth order equation for \( x(t) \) (where for definiteness, I have put \( \mu = 10 \):)

\[ 2x(t)^3x^{(4)}(t) + x(t)x''(t)(3x(t)x''(t) - 4) + 3x'(t)^4 + 4x(t)^2 x^{(3)}(t)x'(t) + x'(t)^2(2 - 12x(t)x''(t)) - 1 = \lambda^* x(t)^4, \] (6.3)

(where \( \lambda^* = \lambda / 41 \)), which should then be solved, subject to the condition (6.1) above.

What I do want to stress is that this equation, even in the case without mass, is too complicated to make an ordinary mathematical theory out of. Once the right framework is formulated, it is of course no problem to solve it numerically. But the point is that the equation (6.3) may contain many different things which are almost impossible to predict from a physical starting point.

Let us consider an example of what kind of results could be obtained. We can simplify the problem somewhat by assuming the function \( x(t) \) to be time-symmetric, i.e. we assume that

\[ x(-t) = x(t). \] (6.4)

This assumption immediately gives that all odd derivatives of \( x(t) \) must vanish at \( t = 0 \). If we therefore let \( x(0) = 1 \) (as a choice of unit) and \( x''(0) = -b \), then together with \( x'(0) = x^{(3)}(0) = 0 \), we have (for a given value of \( \lambda^* \)) enough data to solve equation (6.3) from \( t = 0 \) to \( t = T_0 \) (and by symmetry hence on the full interval \([-T_0, T_0]\)).

In principle, we could now start from “reasonable” values for \( b, T_0 \) and \( V_0 \), try to find a solution of equation (6.3), \( x(t) = x_0(t) \), and then try to adjust the parameter \( \lambda^* \) so as to fulfill (6.1).

In practice however, there seems to be almost no way to estimate the values of \( b, T_0 \) and \( V_0 \) from physical observations, and in addition the equation (6.3) is an very complicated one, which makes it practically very difficult to proceed in this way. Here, we will therefor simply show what kind of solutions can occur for special choices of \( b \) and \( \lambda^* \). Thus, the values of \( T_0 \) and \( V_0 \) which fits with these parameters have only be determined post facto.

The plot of \( x(t) \) in this section uses the values \( \lambda^* = 35.86123 \) and \( b = \frac{1}{2} \), implying values of \( T_0 = 2.38 \) and \( V_0 = 1.93 \). Let us start by looking at what is happening near the Big Bang (figure 3).
expansion, i.e. where $x''(t) > 0$ (x(t) is convex) on intervals which coarsely correspond to about 3% of the whole interval $\left[-T_0, T_0\right]$.

There are a few final points to stress about these computations:

• First of all the model as it stands is not very realistic because it describes at best a universe without mass. And to introduce mass inevitably would require physical ideas which I have no intention to go into in this paper. On the other hand, already this case is sufficiently complex to produce a lot of phenomena which could be relevant for cosmology. The corresponding massless model using ordinary relativity (with or without the cosmological constant) produces only simple and non-interesting models.

• A solution to the Euler-Lagrange equation is not the same thing as a solution to the variational problem. In fact, most of the solutions to equation (6.3) will not correspond to minimizing solutions at all. Again, it is a non-trivial problem for computational mathematics to sort this out.

• The precision of the software used is not enough to draw very firm conclusions about the convexity, in particular since the effect is rather small. But it is clear that the behavior differs considerably from the usual models of cosmology. Also, in view of the singularities in equation (6.3) near the points $-T_0$ and $T_0$, the behavior is uncertain there.

VII. CONCLUSION

In this paper, I have tried to give two examples of important problems in cosmology, where the computational difficulties are so great that it is very difficult to even formulate the right models without involving computational mathematics. It has not been my intention to go into various ways of improving these specific models here. My point is just to stress that such an improvement can only be accomplished through an intimate cooperation between physics and computational mathematics.

REFERENCES


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