Uniform Roe Algebras as Crossed Product

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Abstract—We define what a coarse space is, and we study a number of ways of constructing a coarse structure on a set so as to make it into a coarse space. We also consider some of the elementary concepts associated with coarse spaces. A discrete group $\mathbb{G}$ has natural coarse structure which allows us to define the uniform Roe algebra, $C_0^\ast(\mathbb{G})$. The reduced $C^\ast$ algebra $C_0^\ast(\mathbb{G})$ is naturally contained in $C_0^\ast(\mathbb{G})$. In this paper, we will characterize $C_0^\ast(\mathbb{G})$ as a crossed product.

Index Terms—Invariant Approximation Property, Uniform Roe algebras

I. INTRODUCTION

Uniform Roe $C^\ast$—algebras provide, among other things, a link between coarse geometry and $C^\ast$—algebra theory. Let $\mathbb{G}$ be a discrete group. The reader is referred to Roe [12], Kannan [9], Jolissaint [8], Brown and Ozawa [5], Brodzki [4] and Anantharaman-Delaroche [1] for the details on the invariant approximation property and the coarse geometry. The uniform Roe algebra $C_0^\ast(\mathbb{G})$ is the $C^\ast$—algebra completion of the algebra of bounded operators on $\ell^2(\mathbb{G})$ which have finite propagation. In other words: According to Roe [12] $\mathbb{G}$ has the invariant approximation property (IAP) if

$$C_0^\ast(\mathbb{G}) = C(\mathbb{G}) \rtimes_r \mathbb{G} \cong \ell^\infty(\mathbb{G}) \rtimes_{alg} \mathbb{G}.$$

In section 4, we study the crossed product of $C^\ast$—algebra. In section 5 we study the following statements:

$$C_0^\ast(\mathbb{G}) = C(\mathbb{G}) \rtimes_r \mathbb{G} \cong \ell^\infty(\mathbb{G}) \rtimes_{alg} \mathbb{G}.$$

and we show the induce action of $\mathbb{G}$ on $C(\beta \mathbb{G}) \rtimes_r \mathbb{G}$. The main purpose of this paper is to prove that the Theorem 5.2, in section 5. We also show that the elements of $\ell^\infty(\mathbb{G}) \rtimes_{alg} \mathbb{G}$ which are invariant under $Adp$ are of the form $(\ell^\infty(\mathbb{G}))^{\pi(\mathbb{G})} \rtimes_{alg} \mathbb{G}$.

II. PRELIMINARIES

Coarse geometry is the study of the large scale properties of spaces. The notion of large scale is quantified by means of a coarse structure. First we recall the following definitions:

Definition 2.1: Let $X,Y$ be metric spaces and $f: X \to Y$ a not necessarily continuous map.

1. The map $f$ is called coarsely proper (or metrically proper), if the inverse image of a bounded set is bounded.
2. The map $f$ is called coarsely uniform (or uniformly bornologous), if for every $r \geq 0$ there is $s(r) > 0$ such that for all $x,y \in X$,

$$d(x,y) \leq r \Rightarrow d(f(x),f(y)) \leq s(r)$$

3. The map $f$ is called a coarse map, if it is coarsely proper and coarsely uniform.
4. Let $X$ be a set. Two maps $f,g: X \to X$, are called close if there is $C > 0$ such that for all $x \in X$,

$$s \in S, \quad d(f(s),g(s)) < C.$$

5. A subset $E$ of $X \times X$ is called controlled (or entourage), if the coordinate projection a

$$\pi_1,\pi_1: E \to X$$

is close.

Definition 2.2: A coarse structure on a set $X$ is a collection of subsets of $X \times X$, called the controlled sets or entourages for the coarse structure, which contains the diagonal and is closed under the formation of subsets, inverses, products, and (finite) unions.

It is easy to see that the controlled sets associated to a metric space $X$ have the following properties:

- Any subset of a controlled set is controlled;
- The transpose

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We next recall some definitions about uniform Roe algebra and metric property of a discrete group. Let $X$ be a discrete metric space.

**Definition 2.6:** We say that discrete metric space $X$ has bounded geometry if for all $R$ there exists $N$ in $\mathbb{N}$ such that for all $x \in X$, $|BR(x)| < N$, where

$$B_R(x) = \{x \in X : d(x, y) \leq r\}.$$  

**Definition 2.7:** A kernel $\phi: X \times X \rightarrow \mathbb{C}$

1. is bounded if there, exists $M > 0$ such that $|\phi(s, t)| < M$ for all $s, t \in X$
2. has finite propagation if, there exists $R > 0$ such that $\phi(s, t) = 0$, $d(s, t) > R$

**III. INVARIANT APPROXIMATION PROPERTY**

In this section we will give the definition of invariant approximation property. A discrete group $G$ has a natural coarse structure which allows us to define the uniform Roe algebra $C_0^*(G)$.

A group $G$ can be equipped with either the left or right-invariant of the metric. A choice of one of the determines whether $C_0^*(G) \subseteq C_0^*(G)$ or $A(G)$ is a subalgebra of the uniform Roe algebra $C_0^*(G)$ of $G$ as we now explain. If the metric of $G$ is right-invariant then

$$C_0^*(G) \subseteq C_0^*(G).$$

Let $d_1$ be the right-invariant metric on $G$.  

$$d_1(x, y) = d_1(xg, yg) \forall g \in G.$$  

The operator $\lambda(g)$ is given by the matrix. Let

$$A^2_g(x, y) = \begin{cases} 1 & \text{if } x = yg, \\ 0 & \text{otherwise}. \end{cases}$$

Note that $A^2_g(x, y)$ is right-invariant:

$$A^2_g(xt, yt) = \begin{cases} 1 & xt = ygt, \\ 0 & \text{otherwise}. \end{cases}$$

Therefore: $A^2_g(xt, yt) = A^2_g(x, y)$. If the metric on $G$ is right-invariant, $A^2_g(x, y)$ is of finite propagation and $A^2_g(x, y) \in C_0^*(G)$ since $A^2_g(x, y)$ is non-zero when $y^{-1}x = g$ and so $d_1(x, y) = d_1(xg, yg)$. 

Let $(X, d)$ be a metric space, we say that the metric $d$ induces a coarse structure on $X$, which is called a bounded coarse structure. Coarse geometry is the study of metric spaces (or perhaps more general objects) from a 'large scale' point of view, so that two spaces which 'look the same from a great distance' are considered equivalent.

Let $(X, d)$ be a metric space, we say that the metric $d$ induces a coarse structure on $X$, which is called a bounded coarse structure. More precisely, we can define the bounded coarse structure induced by the metric $d$ as follows: Set

$$D_r := \{(x, y) \in X \times X : d(x, y) < r\}.$$  

Then $E \subseteq X \times X$ is controlled, if $E \subseteq D_r$ for some $r > 0$. The following is an example of coarse structure.

**Example 2.4:** Let $G$ be a finitely generated group. Then the bounded coarse structure associated to any word metric on $G$ is generated by the diagonals

$$\Delta_g = \{(h, hg) : h \in G\},$$

as $g$ runs over $G$. We shall denote the finite propagation kernels on $X$ by $A^2(X)$.

**Definition 2.5:** The uniform Roe algebra of a metric space $X$ is the closure of $A^2(X)$ in the algebra $B(l^2(X))$ of bounded operators on $X$.  

If a discrete group $G$ is equipped with its bounded coarse structure introduced in Example 2.4 then one can associated with it to uniform Roe algebra $C_0^*(G)$ by repeating the above.
Hence any element of $C[G]$ will finite propagation and this assignment extends to an inclusion $C^*_G (G) \hookrightarrow C^*_H (G)$. Similarly we can show that if the metric on $G$ is left-invariant then

$$C^*_G (G) \subseteq C^*_H (G).$$

Let us now choose a right invariant metric for $G$ so that $C^*_G (G) \hookrightarrow C^*_H (G)$. The following important result as given in [9].

**Lemma 3.1:** If $T \in C_G^c (X)$ has kernel $A(x,y)$, then $A \rho_t (T)$ has kernel $A(xt, yt)$.

In general, if $T \in C^c (X)$ then $\forall x, y \in G$, $\langle Ad \rho_t (T) \delta_{x^t}, \delta_y \rangle = \langle T \delta_{x^t}, \delta_y \rangle$.

So the operator $T$ is $Ad \rho_t$ invariant if and only if $\langle Ad \rho_t (T) \delta_{x^t}, \delta_y \rangle = \langle T \delta_{x^t}, \delta_y \rangle, \forall x, y \in X, \forall t \in G$.

We now define the invariant approximation property: (IAP).

**Definition 3.2:** We say that $G$ has the invariant approximation property (IAP) if

$$C^*_G (G)^G = C^*_H (G).$$

**IV. CROSSED PRODUCT OF $C^*$ – ALGEBRAS**

Let $G$ be a discrete group. Let $\alpha: G \rtimes H$ be an action of $G$ on a $C^*$ algebra $H$: $\alpha$ is a homomorphism from group $G$ into the group $Aut(H)$ of automorphisms of $H$. This mean that for each $g \in G$ there is a defined automorphisms $\alpha(g)$ of $H$ given by:

$$\alpha(x) \alpha(y) = \alpha(xy).$$

Any element of the algebraic crossed product of $A$ by $G$ is the formal sum $\sum a_t u_t$, where $u_t$ is unitary, $a_t \in A$ and $t \in G$, and

$$u_t u_{t'} = u_{t+t'}.$$ We denote by $H[G]$ the $C^*$ algebra of formal sums $a = \sum a_t u_t$, where $t \mapsto a_t$ is a map from $G$ into $H$ with finite support and where the operations are given by the following rules:

$$a_t b_s = a a_t (b)ts$$

$$(a)^* = a_{t^{-1}}(a)t^{-1}$$

for $a, b \in H$ and $s, t \in G$.

**Definition 4.1:** A covariant representation of $G \rtimes H$ is a pair $(\pi, \rho)$, where $\pi$ and $\rho$ are unitary representation of $G$ and representation of $H$ in the same Hilbert space $\mathbb{H}$ respectively, satisfying the covariance rule

$$\forall \ a, t \in G, \ \pi(t) \rho(a) \pi(t)^* = \rho a_t(\pi(a)),$$

where $\pi: G \rightarrow U(\ell^2 (G))$ and $\rho: H \rightarrow B(\ell^2 (G))$ and $U(\ell^2 (G))$ is unitary bounded operators.

**Definition 4.3:** The full crossed product of $H \times G$ associated with $\alpha : G \rtimes H$ is the $*$ algebra obtained as the completion of $H[G]$ in the norm

$$\|a\| = \sup\|\pi \times \sigma (a)\|,$$

where $\pi \times \sigma$ runs over all covariant representation of $\alpha : G \rtimes H$.

Next we describe the induced covariant representations.

**Definition 4.5:** Let $G$ be a discrete group. Let $\pi$ be a representation of $\mathbb{H}$ on a Hilbert space $\mathbb{H}_0$ and $\mathbb{H} = (\ell^2 (G), \mathbb{H}_0) = \ell^2 (G) \otimes \mathbb{H}_0$.

We define a covariant representation $(\tilde{\pi}, \tilde{\lambda})$ of $G \rtimes H$ acting on $\mathbb{H}$ by

$$\tilde{\pi}(a) \xi(t) = \pi(\alpha_{t^{-1}}(a)) \xi(t)$$

and

$$\tilde{\lambda}(s) \xi(t) = \xi(s^{-1} t)$$

for all $a \in H$ and $s, t \in G$ and all $\xi \in (\ell^2 (G), \mathbb{H}_0)$.

The covariant representation $(\tilde{\pi}, \tilde{\lambda})$ is said to be induced by $\pi$.

**Definition 4.6:** The reduced crossed product of $H \rtimes G$ is the $*$ algebra obtained as the completion of $H[G]$ in the norm

$$\|a\|_r = \sup\|\tilde{\pi} \times \tilde{\lambda} (a)\|$$
for $a \in H[G]$, where $\pi$ is a representation of $H$.

Next, we show the induced action of $G$ on $C(\beta G) \rtimes_r G$.

V. INDUCED ACTION OF $G$ ON $C(\beta G) \rtimes_r G$.

First we shall describe the following statement:

$$C^*_\pi(G) = C(\beta G) \rtimes_r G \cong \ell^\infty(X) \rtimes_{alg} G.$$ 

Any element of $f \in C(\beta G)[G] \subseteq C(\beta G) \rtimes_r G$ defined by

$$f = \sum f_t t$$

where $f_t \in C(\beta G)$ and $t \in G$. Here:

$$\phi (f) = \sum f_t \theta (f_t)$$

$(x, t) \mapsto f_t(x)$ and $\theta (f)(x, t) = f_t(x)$.

This means that an isomorphism between the $*$- algebra $C(\beta G)[G]$ and the $*$- algebra $C(\beta G \times G)$ of continuous functions with compact support on $\beta G \times G$ given by

$$C(\beta G)[G] = C(\beta G \times G).$$

The operation of the $*$- algebra $C(\beta G \times G)$ on $\beta G \times G$ is given by the following: let $F, G \in C(\beta G \times G)$ then we have

$$(F \ast G)(x, s) = \sum F(x, t)G(t^{-1}x, t^{-1}s)$$

and

$$F(x, s) = F(s^{-1}x, s^{-1}).$$

In addition:

$$F \mapsto F \circ J$$

where $J: (s, t) \mapsto (s^{-1}, s^{-1}t)$ which gives

$$C(\beta G \times G) = \text{bounded kernel with finite propagation on } \beta G \times \beta G.$$ 

The following theorem is from Roe [5].

**Theorem 5.1:** The map between $\pi: f \mapsto \text{Op}(\theta(f) \circ f)$ extends to an isomorphism between the $C^*-$ algebra $C^*_\pi(G) = C(\beta G) \rtimes_r G \cong \ell^\infty(X) \rtimes_{alg} G$.

The uniform Roe algebra, $C^*_\pi(G)$ acts on $\ell^2(G)$, $G$ has unitary representation on $\ell^2(G)$. (e.g. a right regular representation):

$$C^*_\pi(G)^G = \{ T \in C^*_\pi(G): Adp(t) = T \text{ for all } t \in T\}$$

where $p$ is the left regular representation of $G$. Let $T \in C^*_\pi(G)$ and $g \in G$, then we have

$$\forall \rho \in G \text{ Ad}(\rho(g))(T) = p_g T p_g^*.$$ 

We obtained $C(\beta G) \rtimes_r G$, which forms a covariant representation of $\ell^\infty(X) \rtimes_r G$. In Proposition 5.1, we will use $H = \ell^\infty(G)$, and the algebraic crossed product of $H$ by $G$ is the $*$- algebras.

**Theorem 5.2:** Assume that the algebraic crossed product $\ell^\infty(X) \rtimes_{alg} G$ is given by:

- the pointwise action of $\ell^\infty(G)$ on $\ell^2(G)$

$$(a\xi)(s) = a(s)\xi(s), a \in \ell^\infty(G), \xi \in \ell^2(G),$$

- the left regular representation $\lambda$ of $G$ on $\ell^2(G)$

$$(\lambda(g)\xi)(s) = \xi(gs),$$

Then the conjugation action induced by the right regular representation $p$ of $G$ on $\ell^2(G)$

$$[(p(g)\xi)(s) = \xi(g^{-1}s)] \text{ works by }$$

$$\rho(h)a\lambda(g)p(h)^* = (p(h)a)\lambda(g)$$

where $(p(h)a)(s) = a(hs)$.

**Proof:** Let $\xi \in \ell^2(G)$ and $g, h \in G$. Then

$$(p(h)a\lambda(g)p(h)^*\xi)(s) = (p(h)ap(h)^*\lambda(g)\xi)(s) = (p(h)ap(h)^*\xi)(g^{-1}s) = (p(h)a)(g^{-1}s)\xi(g^{-1}s) = (p(h)a)(g^{-1}s)\xi(g^{-1}s) = (p(h)a)\lambda(g)\xi(s).$$

We also have

$$\rho(h)a\lambda(g)p(h)^* = a\lambda(g).$$

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Next, we describe that conclusion of the above result (Theorem 5.2).

**Conclusion 5.3:** It seems that the elements of $\ell^\infty(G) \rtimes_{\text{alg}} G$ which are invariant under $\text{Ad}\rho$ are of the form

$$\ell^\infty(G)\rtimes\rho(G) \rtimes_{\text{alg}} G.$$ 

Therefore any element of $a\lambda(g)$, where $a \in \ell^\infty(G)$ and $g \in G$.

**REFERENCES**


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