The CATASSTROPHE MAP: PROPERTIES of a PRODUCTION MODEL with UNCERTAINTY

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***Abstract* –The paper generalizes the natural projection approach introduced by Balasko [4] for study of the qualitative equilibrium structure of exchange economies to a two period private ownership production model with uncertainty. It shows that long run equilibrium properties of the production model are those of the pure exchange economy with production adjusted demand functions. Associated with every long run equilibrium there exist a finite, odd number of short run equilibria.**

***Index Terms* –Existence of Equilibrium, Uncertainty, Production**

I. INTRODUCTION

Balasko [2] shows that comparative static analysis of the Debreu mapping amounts to a qualitative study of the restriction of the projection $(p,\omega) \mapsto \omega$ to the equilibrium manifold $\mathcal{E}$. A summary of results on qualitative properties of the equilibrium set for exchange economies and economic applications based upon the natural projection approach are found in Balasko [3]. The natural projection π is the mathematical tool used to study the structure of the set $\mathcal{E}$, the set of solutions of the equilibrium equation $z(p,\omega) = 0$ (aggregate excess demand function), for varying parameters $\omega \in \Omega$. Economic equilibrium properties do not only depend on the structure of $\mathcal{E}$ but also on how this set is embedded in the Cartesian product defined by the set of prices $\mathbf{S}$ and the set of economies Ω. These properties are derived from restricting π to $\mathcal{E} \subset \mathbf{S} \times \Omega$, a mapping from $\mathcal{E}$ into the set of economies Ω.

For example, Balasko shows that for the static Arrow-Debreu model and for the two period exchange model (Balasko and Cass, [5]) that existence of competitive equilibrium is a consequence of the projection mapping being smooth and proper. Its inverse defines a ramified covering with a finite set of layers for regular economies. The number of equilibria is not only finite but always odd and constant for some sections of the parameter space Ω. Another remarkable property of the natural projection approach follows immediately from its relation to the Walras correspondence $W(\omega) \times \{\omega\} = \pi^{-1}$ [1], [10].

Originally, the structure of the equilibrium set $\mathcal{E}$ following the natural projection approach is studied in the context of static exchange models. This set up does not consider for the many situations where production is the center object of study. This paper considers the extension of the natural projection approach to the study of economic equilibrium properties of a two period production model with uncertainty. It is shown that some properties of the solution set of the equilibrium equation of the exchange model generalize to the smooth long run production model with convex production sets. This essentially follows from the fact that every equilibrium of the two period production model is also an equilibrium of the exchange model with production adjusted demand functions.

Section 2 introduces the long run model of production. It shows that every long run equilibrium of the two period production model with uncertainty is an equilibrium of the exchange model with production adjusted demand functions. Section 3 explores the equilibrium structure of the long run private ownership production model. It generalizes the natural projection approach to economies with production and uncertainty. Section 4 considers the full model of production where firms chose long and short run profit maximizing activities. It shows that long run equilibria always exist and that the number of short run equilibria associated with every long run equilibrium is finite and odd. Section 5 is a conclusion.

II. THE LONG RUN PRIVATE OWNERSHIP PRODUCTION MODEL WITH UNCERTAINTY

Consider the two period private ownership production model $P(L)$ introduced in Debreu [7], chapter 7. Uncertainty is denoted by a realization of a random variable s in the set of mutually exclusive and exhaustive states of nature denoted by $s \in \{1, \dots, S\}$. There are $i \in \{1, \dots, m\}$consumers, $j \in \{1, \dots, n\}$ producers, and $k \in \{1, \dots, l\}$ physical goods. For all consumers$i \in \{1, \dots, m\}$, a consumption bundle is a collection of vectors$x_i = (x_i(0), \dots, x_i(s), \dots, x_i(S)) \in X_i = \mathbb{R}^{l(S+1)}_{++}$, where consumption in a particular sate $s \in \{0,1, \dots, S\}$ is a vector$x_i(s) = (x_i^1(s), \dots, x_i^l(s)) \in \mathbb{R}^l_{++}$. Associated with physical commodities is a set of normalized prices, denoted$S = \left\{p \in \mathbb{R}^{l(S+1)}_{++} : p^l(s) = 1, \forall_s \in \{0,1, \dots, S\}\right\}$. For a particular realization $s \in \{1, \dots, S\}$ denote the state price vector$p(s) \in S = \mathbb{R}^{(l-1)}_{++} \times \{1\}$. Consumers are further endowed with a fraction θ_{ij} representing the exogenously determined ownership structure of the private ownership production economy. θ_{ij} satisfied for each $j \in \{1, \dots, n\}$ and $i \in$

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$\{1,\dots,m\} 0 \leq \theta_{ij} \leq 1$, and $\sum_i \theta_{ij} = 1$. Denote the set of ownership structures $\Theta = \{\theta_{ij} \in \mathbb{R}^{nm}_{+} : \sum_i \theta_{ij} \sum_i = 1, \forall_i \in \{1,\dots,m\}\}$.

Consumers are endowed with initial resources $\omega_i = (\omega_i(0),\dots,\omega_i(s),\dots,\omega_i(S)) \in \Omega_i = \mathbb{R}^{l(S+1)}_{++}$, where initial endowments in a particular state $s \in \{0,1,\dots,S\}$ is a vector $\omega_i(s) = (\omega_i^1(s),\dots,\omega_i^l(s)) \in \mathbb{R}^l_{++}$. Consumer $i \in \{1,\dots,m\}$ is further characterized by a smooth Marschallian demand function $f_i : S \times \mathbb{R}^{S+1}_{++} \to \mathbb{R}^{l(S+1)}_{++}$, where $f_i(p, w_i)$ is defined for price vector $p \in S$ and wealth level $w_i \in \mathbb{R}^{S+1}_{++}$, Debreu [8].

Producers are characterized by production sets and their smooth supply functions. The main property of the long run production model is that all activities of the firm are variable. An activity y_j is a collection of vectors $y_j = (y_j(0),\dots,y_j(s),\dots,y_j(S)) \in \mathbb{R}^{l(S+1)}$, where an activity in state $s = 0$ is a vector of inputs $y_j(0) = (y_j^1(0),\dots,y_j^l(0)) \in \mathbb{R}^l_{-}$, and $y_j(s) = (y_j^1(s)) \in \mathbb{R}^l_{+}$ is associated vector of outputs in state $s \in \{1,\dots,S\}$. Let $\xi_j : S \to \mathbb{R}^{l(S+1)}$ denote the supply function of firm $j \in \{1,\dots,n\}$, where $\xi_j(p)$ is defined on the set of normalized prices. Standard assumptions of smoth production economies introduced in [7] hold for each production set Y_j. In particular Y_j is convex, inactivity 0 is an element in Y_j, and the efficient boundary of Y_j has a strictly positive Gaussian curvature.

2.1. Equilibrium $P(L)$

Each consumer $i \in \{1,\dots,m\}$ chooses a utility maximizing consumption bundle $x_i \in X_i$ a fixed $\omega_i \in \Omega$ and $\theta_{ij} \in \Theta$ satisfying his budget constraints. Each producer $j \in \{1,\dots,n\}$ chooses profit maximizing net activities $y_j \in Y_j$ at competitive prices $p \in S$.

Definition 1 *An equilibrium of the two period private ownership production model with uncertainty is a price vector $p \in \mathbf{S}$, at fix pair $(\omega, \theta) \in \Omega \times \Theta$ if for utility maximizing consumers $i \in \{1,\dots,m\}$ and profit maximizing producers $j \in \{1,\dots,n\}$ the excess demand function $z(p,\omega) = 0$ defined by*

$$\sum_i f_i\left(p, p \cdot \omega_i + \sum_j \theta_{ij} p \cdot \xi_j(p)\right) = \sum_i \omega_i + \sum_j \xi_j(p).$$

is satisfied.

An equilibrium allocation is a pair $(x, y) \in \mathbb{R}^{l(s+1)m}_{++} \times \mathbb{R}^{l(s+1)n}$ associated with an equilibrium price vector $p \in S$ for fixed parameters $(\omega, \theta) \in \Omega \times \Theta$. There are $l(S+1)$ equilibrium equations less $(S+1)$ equations satisfying Walras' law $p \Box z(p,\omega) = 0$, hence we have a system of $l(S+1) - (S+1)$ linearly independent equations1 . This amounts to the number of unknowns, given the number of normalized prices of $(S+1)$ A study of the qualitative equilibrium structure of the two period private ownership production model with uncertainty amounts to a study of the structure of the solution set of the equilibrium equation $z(p,\omega) = 0$. The first result is an equivalence relation between the two period exchange model with uncertainty and the two period production model with uncertainty. The relation between these models follows from the definition of a two period exchange model with production adjusted Marshallian demand functions.

Let $\zeta_i(p) = \sum_j \theta_{ij} \xi_j(p)$ for any price system $p \in S$. Let $h_i : S \times \mathbb{R}^{S+1}_{++} \to \mathbb{R}^{l(S+1)}_{++}$ defined by $h_i(p, w_i) = f_i(p, w_i + p \cdot \zeta_i(p)) - \zeta_i(p)$ denote the demand function of the two period "production adjusted" exchange model, where for every $i \in \{1,\dots,m\}$ ownership structure θ_{ij} is fixed, and total wealth defined by $p \cdot \omega_i + p \cdot \zeta_i(p)$. Now, consider equilibrium equation (1) of the production model given by

$$\sum_i f_i\left(p, p \cdot (\omega_i + \zeta_i(p))\right) = \sum_i \omega_i + \sum_j \xi_j(p).$$

Rewriting the supply function in terms of ownership structure and summing over $i \Sigma_j \Sigma_i \theta_{ij} \xi_j(p)$, and using definition $\zeta_i(p) = \Sigma_j \theta_{ij} \xi_j(p)$ yields

$$\sum_i f_i\left(p, p \cdot (\omega_i + \zeta_i(p))\right) = \sum_i \omega_i + \sum_j \zeta_i(p).$$

which can be rewritten into

$$\sum_i f_i\left(p, p \cdot (\omega_i + \zeta_i(p))\right) - \sum_j \zeta_i(p) = \sum_i \omega_i,$$

hence by definition of production adjusted demand functions obtain

$$h_i(p, w_i) = f_i\left(p, w_i + p \cdot \zeta_i(p)\right) - \zeta_i(p).$$

We have proved the equivalence between the two period exchange model and the long run production model by showing that the production model can be reformulated in terms of an exchange model with production adjusted demand functions.

Proposition 1 *For fixed $\theta \in \Theta$, $(p,\omega) \in S \times \Omega$ is an equilibrium of the long run production model with uncertainty if and only if $(p,\omega) \in S \times \Omega$ is an equilibrium of the two period exchange model with uncertainty and production adjusted demand functions.*

Next section studies properties of the exchange model with production adjusted demand functions, $P(E)$. We use proposition (1) to derive corollaries for the long run production mode l$P(L)$.

[1] □is the mathematical symbol for the box product, a state by state inner product operation.

3 EQUILIBRIUM STRUCTURE $\mathcal{E}$ *of* $P(E)$ *and* $P(L)$

Let $\mathcal{E}$ denote the set of equilibrium solutions of the production adjusted exchange model $\mathrm{P(E)}$ or the set of solutions of the long run production model $\mathrm{P(E)}$. 2This set consists of pairs $(p,\omega)\in S\times\Omega$ satisfying the equilibrium equations $z(p,\omega)=0$. Formally

$$\mathcal{E}=\left\{(p,\omega)\in S\times\Omega:\sum_i f_i\left(p,p\cdot\omega_i+\sum_j\theta_{ij}p\cdot\xi_j(p)\right)=0\right\}$$

Theorem 1 *The set* $\mathcal{E}$ *of model* $\mathrm{P(E)}$ *is a closed subset of the Euclidean space defined* by $S\times\Omega$.

Proof. $\mathcal{E}$ is defined by pairs $(p,\omega)\in S\times\Omega$ satisfying equilibrium equation (1). $\mathcal{E}$ is the preimage of the vector $0\in\mathbb{R}^{l(S+1)}$ by the smooth mapping $(p,\omega)\mapsto\sum_i f_i\left(p,p\cdot\omega_i+\sum_j\theta_{ij}\,p\cdot\xi_j(p)\right)-\left(\sum_i\omega_i+\sum_j\xi_j(p)\right)=0$ and by the closed map lemma closed [11]. ■

Corollary 1 *The set* $\mathcal{E}$ *of model* $\mathrm{P(L)}$ *is a closed subset of the Euclidean space defined by* $S\times\Omega$.

Proof. Same as above. Note that in both cases continuity of the mapping $(p,\omega)\mapsto\sum_i f_i\Big(p,p\cdot\omega_i+$ jθijp·ξjp−iωi+jξjp=0 is sufficient. Indeed this requires demand functions to be continuous only. ■

Theorem 2 *The set* $\mathcal{E}$ *of model* $\mathrm{P(E)}$ *is a smooth manifold of dimension* $(S+1)lm$.

Proof. Consider the mapping $Z{:}\times\Omega$ into $\mathbb{R}^{l(S+1)}$defined by the smooth mapping $(p,\omega)\mapsto\sum_i f_i\Big(p,p\cdot\omega_i+\sum_j\theta_{ij}\,p\cdot$ ξjp−iωi+jξjp. By theorem (1) $\mathcal{E}$ is the preimage of $0\in\mathbb{R}^{l(S+1)}$. We need to prove that this mapping does not contain critical points. This follows by showing that the linear tangent map DZ is onto. The onto property follows directly from the rank property of the Jacobian matrix chosen for any arbitrary individual $i\in\{1,\dots,m\}$ and state of nature $s\in\{0,1,\dots,S\}$. By the chain rule, we obtain

$$\begin{pmatrix}\frac{\partial f_i^1(s)}{\partial w_i(s)}p^1(s)-1 & \dots & \frac{\partial f_i^1(s)}{\partial w_i(s)}p^{l-1}(s) & \frac{\partial f_i^1(s)}{\partial w_1(s)}\\ \vdots & & & \vdots\\ \frac{\partial f_i^{l-1}(s)}{\partial w_i(s)}p^1(s) & & \frac{\partial f_i^{l-1}(s)}{\partial w_1(s)}p^{l-1}(s)-1 & \frac{\partial f_i^{l-1}(s)}{\partial w_i(s)}\end{pmatrix}.$$

By simple algebraic manipulation we obtain

$$\begin{pmatrix}-1 & & \frac{\partial f_i^1(s)}{\partial w_1(s)}\\ & \ddots & \\ & -1 & \frac{\partial f_i^{l-1}(s)}{\partial w_i(s)}\end{pmatrix}.$$

From which we extract the information required. Rank DZ is equal to $(l-1)$ in every state $s\in\{0,1,\dots,S\}$. By the regular value theorem $\mathcal{E}$ is a smooth manifold parameterized by smooth coordinate functions $\omega=(\omega(0),\dots,\omega(s),\dots,\omega(S)\in\Omega)$. Its dimension is equal to the dimension of $S\times\Omega$ minus $l(S+1)$, hence $\dim(\mathcal{E})=\big((l-1)(S+1)\big)$. ■

Corollary 2 *The set* $\mathcal{E}$ *of model* $\mathrm{P(L)}$ *is a smooth manifold of dimension* $(S+1)lm$.

Proof. Follows along the same lines of the proof above and by applying proposition (1). ■

Following theorem illustrates other economically interesting global properties of the equilibrium manifold. It says that by construction of a diffeomorphism ϕ restricted to the equilibrium manifold $\mathcal{E}$ into $\mathbb{R}_{++}^{l(S+1)}$ that $\mathcal{E}$ is diffeomorphic to the sphere $\mathbb{R}_{++}^{l(S+1)}$implying that the equilibrium manifold is arc-connected, simply connected, and contractible. In order to prove this result, we state a mathematical result (without proof) that we make use of.

Let $f{:}X\to Y$ and $g{:}Y\to X$ be two smooth mappings between smooth manifolds such that $f\circ g{:}Y\to Y$ is the identity mapping Id. Then $Z=g(Y)$ is a smooth sub manifold of X diffeomorphic to Y.3

Theorem 3 *The smooth equilibrium manifold* $\mathcal{E}$ *of model* $\mathrm{P(E)}$ *is diffeomorphic to* $\mathbb{R}_{++}^{l(S+1)}$.

Proof. Let $g{:}S\times\mathbb{R}_{++}^{(S+1)m}\times\mathbb{R}_{++}^{(l-1)(S+1)(m-1)}\to S\times\Omega$ denote a smooth map defined by $\left(p,\overline{\omega}_1,\omega_1^l,\dots,\overline{\omega}_{m-1},\omega_{m-1}^l,\overline{\omega}_m\right)$, and let $f{:}S\times\Omega\to S\times\mathbb{R}_{++}^{(S+1)m}\times\mathbb{R}_{++}^{(l-1)(S+1)(m-1)}$ denote a smooth map defined by $f(p,\omega_1,\dots,\omega_m)=\left(p,p\cdot\omega_1,\dots,p\cdot\omega_p\cdot\overline{\omega}_1,\dots,\overline{\omega}_{m-1}\right)$, where

$$\omega_i^l=w_1-\left(\sum_{l=1}^{l-1}p^l\cdot\omega_i^l\right),\forall_i\in\{1,\dots,m-1\}$$

and

$$\omega_m=\textstyle\sum_{i=1}^m f_i\,(p,w_i)-\sum_{1=1}^{m-1}\omega_i.$$

The strategy of the proof is to apply above lemma. For that need to show that $\mathcal{E}$ is the image of the mapping g, then we can apply above lemma to the mapping $f\circ g$.

Now, to show that (i) $\mathrm{Im}(g)\subset\mathcal{E}$, take $x=(p,\omega_1,\dots,\omega_m,\overline{\omega}_1,\dots,\overline{\omega}_{m-1})$. Next, compute the inner product of (3) with p and apply Walras' law to obtain $\omega_m=p\cdot\omega_m$. From that a

[2] $\mathcal{E}$ is always understood from the context.

[3] See Bourbaki for a proof of this lemma [6].

reformulation of (3) readily follows

$$\sum_i f_i\left(p, p\cdot\omega_i + \sum_j \theta_{ij}\, p\cdot\xi_j(p)\right) = \sum_i \omega_i + \sum_j \xi_j(p),$$

which is the the equilibrium equation (1), hence $\mathrm{Im}(g) \subset \mathcal{E}$. Next, need to show that (ii) $\mathcal{E} \subset \mathrm{Im}(f)$. Take$(p,\omega) \in \mathcal{E}$. It is then trivial to do the computations proving following equality

$$f \circ g(p,\omega) = (p,\omega)$$

from which it readily follows that $\mathcal{E} \subset \mathrm{Im}(f)$. Clearly we have constructed the two smooth relations such that

$$f \circ g = \mathrm{Id},$$

where Id is the identity map defined on $\left(S \times \mathbb{R}_{++}^{(S+1)m} \times \right.$ $\mathbb{R}++l-1S+1m-1$. The

result then follows immediately from above lemma. ■

Corollary 3 *The smooth equilibrium manifold* $\mathcal{E}$ *of model* $P(L)$ *is diffeomorphic to* $\mathbb{R}_{++}^{l(S+1)}$.

Proof. The proof follows immediately from theorem above and proposition (1). The dimension of the sphere is the same as for the$P(E)$ model. The proof is therefore omitted. ■

It remains to be shown that equilibria in the long run production model with uncertainty always exist. The strategy of the proof is to show that the natural projection mapping $\pi: \mathcal{E} \to \Omega$ is smooth and proper. Existence of long run equilibria of the production model with uncertainty follows immediately from lemma (1) and lemma (2) below.

Theorem 4 *Equilibria of the two period production model* $P(L)$ *with uncertainty always exist.*

Lemma 1 $\pi: \mathcal{E} \to \Omega$ *is smooth.*

Proof. Recall that $\mathcal{E}$ is a smooth submanifold of $S \times \Omega$. It follows from the definition of a smooth manifold that its natural embedding $\overline{\pi}: \mathcal{E} \to S \times \Omega$ is itself smooth. The projection mapping $\overline{\pi}: S \times \Omega \to \mathcal{E}$ being itself smooth, it follows that π the restriction of the natural projection to $\mathcal{E}$ as the composition of two smooth mappings $\pi = \overline{\pi} \circ \widehat{\pi}$ is therefore smooth. ■

If X and Y are topological spaces, a map $f: X \to Y$ is said to be proper if for every compact set $K \subset Y$, the inverse image$f^{-1}(K)$ is compact. A sufficient condition for a map to be proper is therefore equivalent to showing that K is compact [6].

Lemma 2 $\pi: \mathcal{E} \to \Omega$ *is proper.*

Proof. Pick an arbitrary ω_i for $i \in \{1, \dots, m\}$. Let $\omega_i \in K_i$ be an element in a compact set K_i. Compactness implies that K_i is bounded from below and from above, hence there exist elements $\omega_i' \le \omega_i \le \omega_i''$. Now, for every $p \in S$ and $\omega_i \in K_i$ need to show (i) that $f(p, w_i)$ is bounded from below. It follows from the definition of $f(p, w_i)$ that

$$u_i(\omega_i) \le u_i\left(f_i(p, w_1)\right)$$

and by non-satiation have also

$$u_i(\omega_i') \le u_i(\omega_1)$$

which by monotonicity of u_i implies that

$$u_i(\omega_i') \le u_i\left(f_i(p, w_1)\right).$$

clearly, there exists some $x_i' \in \mathbb{R}_{++}^{l(S+1)}$ for every $p \in S$ and $\omega_i \in K_i$ satisfying

$$x_i' \le u_i\left(f_i(p, w_1)\right)$$

by boundedness of indifference mappings from below for every $i \in \{1, \dots, m\}$. (ii) We now show that for every $p \in S$ and $\omega_i \in K_i$, $f_i(p, w_1)$ is also bounded from above. For (p, ω_i) have

$$\left(f_i(p, w_1)\right) = \sum_i \omega_i - \sum_{-i} f_i(p, w_1)$$

where

$$\sum_i \omega_i - \sum_{-i} f_i(p, w_1) \le \sum_i \omega_i - \sum_{-i} x_i'$$

Clearly, $f_i(p, w_1)$, is bounded above by some $x_i'' \in \mathbb{R}_{++}^{l(S+1)}$, since for $(p,\omega) \in \mathcal{E}$, $\sum_{-1} \omega_i$ is bounded from above for every $\omega \in K$. Hence have established upper and lower bounds defining a compact set

$$\{x_i' \le f_i(p, w_1) \le x_i''\}$$

for every $(p,\omega) \in \pi^{-1}(K)$. Let G be a compact set defined by the preimage of the diffeomorphism $f_i(p, w_1)$ projected onto S. Now, by continuity of $\pi: \mathcal{E} \to \Omega$, $\pi^{-1}(K)$ is closed in $\mathcal{E}$, which by theorem (1) is a closed subset of $S \times \Omega$. Closedness of $\pi^{-1}(K)$ follows from closedness of $\pi^{-1}(K) \cap G \times K \subset G \times K$. ■

The number of equilibria of the long run production model with uncertainty is odd for any regular economy $\omega \in \Omega$.

Proposition 2 *The modulo 2 degree of* π *is*$+1$.

Proof. For any regular $\omega \in \Omega$ oddness follows immediately from the definition of intersection theory modulo 2 degree. ■

We now define a subset of points on $\mathcal{E}$ at which pairs $(p,\omega) \in \mathcal{E}$ are not regular. Singular points are points associated with the coordinate system of the natural projection map π, at which the rank of the Jacobian matrix is strictly less that $l(S+1)m$.

Definition 2 *The set* $\mathcal{E}_c$ *consists of critical equilibria* $(p,\omega) \in \mathcal{E}$ *defined by the critical points of* π.

$\mathcal{E}_c = \{$ all critical equilibria $(p,\omega) \in \mathcal{E}: (p,\omega) \in \mathcal{E}$ defined by the critical points of π. Following result shows that this set is closed.

Proposition 3 *$\mathcal{E}_c$ is closed.*

Proof. A necessary and sufficient condition for an equilibrium pair$(p,\omega) \in \mathcal{E}$ to be critical is that the determinant of the Jacobian matrix of π, denoted $\det(D\pi)$ is equal to zero. Now, the set of critical values $\mathcal{E}_c$ defined by the preimage of $0 \in \det(D\pi)$ is closed by the closed mapping lemma [11]. Clearly, π, $D\pi$, and the coefficients of $\det(D\pi)$ are all continuous from which the result follows. ■

Definition 3 $\Sigma = \{\omega \in \Omega: \text{for all } \omega \in \Omega \text{ critical values of image of } \pi\}$.

A singular value $\omega \in \Omega$ is the image of π of a critical point $(p,\omega) \in \mathcal{E}_c$ into Ω. The set of regular values is defined by $R = \{\omega \in \Omega: \text{for all } \omega \in \Omega \text{ regular values of image of } \pi\}$. It follows that $R = \Omega \setminus \Sigma$ represents the sets of regular economies. The next proposition states the Σ is closed and of measure zero. This means that the probability of observing an economy with this property is "close" to zero. Hence, its complement R is an open dense set.

Proposition 4 *The set of singular economies Σ is closed and of Lebesgue measure zero in Ω.*

Proof. The proof follows from the application of Sards's theorem which describes the set of singular values of a smooth mapping having the property of Lebesgue measure zero. Hence know that Σ is a set of Lebesgue measure zero. Closedness of Σ follows from the properness of π. To see this recall that Σ is the image of π for pairs $(p,\omega) \in \mathcal{E}_c$ is closed. This follows from proposition (3). The property that Σ is a closed set follows from lemma (2). ■

4. ORGANIZATION OF PRODUCTION IN THE LONG AND SHORT RUN

4.1 *The general model of the firm*

Consider a version of the private ownership production model with uncertainty introduced above. The diffirenece to the model above follows from the additional structure imposed on the production set available to the firm. In$P(L)$ firms choose profit maximizing net inputs in $t = 0$ with associated outputs in $t = 1$. For example

$$Y_j = \{(y(0), y(1)) \in \mathbb{R}^l_- \times \mathbb{R}^{lS}_+ : F(y(0), y(1)) \leq 0\}.$$

In the new long run production model$P(L)$ firms choose long run factors of production such as capital in period$t = 0$. The total amount of capital purchased by a firm in $t = 0$ determines the maximal units of production a firm can produce in period $t = 1$, called production capacity, $\overline{K}$measured in units of outputs $y^k_j(1) \geq 0$for some$k \in \{1,\dots,l\}$. Once production capacity is installed, actual production of goods takes place in period $t = 1$ where the firm's problem is to choose profit maximizing short run net activities with labor as a typical example of a short run input of production. For example

$$Y_j = \{(y(0), y(1)) \in \mathbb{R}^l_- \times \mathbb{R}^{aS}_- \times \mathbb{R}^{bS}_+ : F(y(0), y(1)) \leq 0\},$$

where$a + b = l$, $y_j(0) \leq 0$, and $y^k_j(1) \geq 0$ for some $k \in \{1,\dots,l\}$ and $y^k_j(1) < 0$ for remaining $k \in \{1,\dots,l\}$. In the model$P(L)$ long and short run activities of the firm are variable. Associated with every long run production model$P(L)$ there exist a short run production model$P(S)$. The main property of the short run model is that the production set available to a firm is $Y_j(\overline{K})$. For example

$$Y_j(\overline{K}) = \{(\bar{y}(0), y(1)) \in \mathbb{R}^l_- \times \mathbb{R}^{aS}_- \times \mathbb{R}^{bS}_+ : F(\bar{y}(0), y(1)) \leq 0\}.$$

The objective of consumers is to maximize utilities subject to a sequence of budget constraints. In the long run, each consumer maximizes utility from consumption goods over both periods. In the short run, each consumer maximizes utility at the realized state of the world and short run consumption constraints. We apply the same methodology of the previous section by introducing production adjusted demand functions. Let $h_i: S \times \mathbb{R}^{S+1}_{++} \to \mathbb{R}^{l(S+1)}_{++}$ defined by $h_i(p, w_i) = f_i\left(p, w_i + p \cdot \zeta_i(p)\right) - \zeta_i(p)$ denote the Marschalian demand function of the two period "production adjusted" exchange model, where for every $i \in \{1,\dots,m\}$ ownership structure θ_{ij} is fixed, and total wealth defined by $p, \omega_i + p \cdot \zeta_i(p)$. Formally, every $i \in \{1,\dots,m\}$

$$(x_i) \in \operatorname{argmax}\{u_i(x_i): x_i \in B\}$$

where

$$B = \left\{ p(s) \cdot (x_i(s) - \omega_i(s)) = \sum_j \theta_{ij} p(s) \cdot y_j(s), \forall_s \in \{1,\dots,m\} \right\}$$

The problem of the first is to maximize profits subject to a sequence of constraints. In the long run production model$P(L)$ all net activities $(y(0), y(1))$ in Y_j are variable. For every $j \in \{1,\dots,n\}$ profit maximization is formally defined by

$$(y_j) \in \operatorname{argmax}\{p \square y_j: y_j \in Y_j, \forall_s \in \{0,1,\dots,S\}\}$$

where the box product $\square$is a state by state inner product operation on the price vector $p(s)$ and activity $y_j(s)$ for $s = 0$ in period $t = 0$, and $s \in \{1,\dots,S\}$ in period $t = 1$.

In the short run production model $P(S)$ each producer $j \in \{1, \dots, n\}$ chooses profit maximizing short run net activities $y_j(1)$ in $Y_j(\bar{K})$ at fixed production capacity level $\bar{K}$ determined in period $= 0$. Formally every $j \in \{1, \dots, n\}$

$$\left(y_j(s)\right) \in \operatorname{argmax}\left\{p\,(s) \cdot y_j(s) : y_j \in Y_j(\bar{K}), \forall_s \in \{1, \dots, S\}\right\}$$

The equilibrium equations $z(p, \omega) = 0$ for this model are:

$$\sum_i f_i\left(p, p \cdot \omega_i + \sum_j \theta_{ij} p \cdot \xi_j(p)\right) = \sum_i \omega_i + \sum_j \xi_j(p).$$

Proposition 5 *For fixed* $\theta \in \Theta$ *and* $\bar{K}$, $(p, \omega) \in S \times \Omega$ *is an equilibrium of the short run production model* $P(S)$ *if and only if* $(p, \omega) \in S \times \Omega$ *is an equilibrium of the production adjusted exchange model.*

This is essentially the result of proposition (1) adjusted for fixed production capacity. The proof is therefore omitted. It goes along the same lines of the proof of proposition (1). In addition we note that $P(S) \subset P(L)$.

Theorem 5 π^{-1} *restricted to the short run production model* $P(S)$ *is a finite covering for every* $\omega \in R$.

Proof. Let $\{p\}$ consist of a single element of $\pi^{-1}(\omega)$. Consider the tangent map of elements of $\mathcal{E}$ not contained in the set of critical points, $p \notin \mathcal{E}_c$. Then as a non-critical point in $\mathcal{E}$ there exists a bijective map $D_{\pi p}$ which by the inverse function theorem implies that $\pi: \mathcal{E} \longrightarrow \Omega$ is locally a diffeomorphism. By the inverse function theorem there exists an open set U of $\omega \in R$ and an open set V of $p \in \mathcal{E}$ such that the restriction of the natural projection to V, $\pi|\,v: V \rightarrow U$ is a diffeomorphism. It follows from the one-to-one property of this map that $\pi^{-1}(\omega) \cap V = \{p\}$. Since V is open in $\mathcal{E}$ it follows from the definition of open sets of $\pi^{-1}(p)$ as intersections with $\pi^{-1}(\omega)$ of open sets of $\mathcal{E}$ that the subset $\{p\}$ is open in $\pi^{-1}(p)$. The union of all open subsets $\{p\} \in \pi^{-1}(\omega)$ define an open covering P of $\{p\} \in \pi^{-1}(\omega)$. Compactness of the set $\pi^{-1}(\omega)$ follows from compactness of the preimage of a compact set $\{\omega\}$ by the proper mapping $\pi: \mathcal{E} \longrightarrow \Omega$. It follows from compactness of $\pi^{-1}(\omega)$ that the open covering has a finite subcovering defined by the unique element of $\pi^{-1}(\omega)$. The union of a finite number of elements defines the set $\pi^{-1}(\omega)$ which is therefore a finite set. This proves finiteness of the number of equilibria. ■

Theorem 6 *For every regular* $\omega \in R$ *restricted to* $P(S)$ *there exists an open neighborhood* $U \subset R$ *of* ω. *For every nonempty* $\pi^{-1}(\omega)$, $\pi^{-1}(U)$ *is the union of a finite number of pairwise disjoint open sets* $V_1, \dots, V_n$ *and the restriction of the map* π *defined by* $\pi_k: V_k \rightarrow U$ *being a diffeomorphism for* $k \in \{1, \dots, n\}$.

Proof. By theorem (5) have a nonempty finite set of elements defined by $\pi^{-1}(\omega)$. Let $p_1, \dots, p_n$ be all elements of the inverse image of $\pi: \mathcal{E} \longrightarrow \Omega$ defined by $\pi^{-1}(\omega)$ for every $\omega \in R$. Provided that all open sets are small enough, it is always possible to consider open disjoint unions $\bar{U}_1, \dots, \bar{U}_n$ in $\mathcal{E}$ of $p_1, \dots, p_n$ such that $\pi|U_I$ where $U_i = \pi(\bar{U}_I)$ is a diffeomorphism. $\mathcal{E} \setminus (\bar{U}_1 \cup, \dots, \cup$ Un is closed in $\mathcal{E}$ and its image by properness of π is closed in Ω. Let $U = (U_1 \cap, \dots, \cap U_n)\pi\big(\mathcal{E} \setminus (\bar{U}_1 \cup, \dots, \cup$ Un. Obviously, U is open in Ω. We need to show that $\omega \in U$ follows from $\pi^{-1}(\omega) \subset \bar{U}_1 \cup, \dots, \cup \bar{U}_n$ implying that $\omega \in U$ does not belong to $\pi\big(\mathcal{E} \setminus (\bar{U}_1 \cup, \dots, \cup \bar{U}_n)\big)$. Let $V_n = \bar{U}_n \cap \pi^{-1}(U)$. Then for all $k \in \{1, \dots, n\}$, $\pi_k|V_k$ obviously determines a diffeomorphism between V_n and $\pi(V_n)$. It only remains to prove that $\pi^{-1}(U)$ is equal to the union of all V_n. This follows by contradiction. Let $\{p\} \in \pi^{-1}(U)$. Assume that $\{p\}$ does not belong to any V_n. Then $\{p\}$ must belong to $\mathcal{E} \setminus (\bar{U}_1 \cup, \dots, \cup \bar{U}_n)$, implying that $\omega = \pi(p) \in \pi\big(\mathcal{E} \setminus (\bar{U}_1 \cup, \dots, \cup \bar{U}_n)\big)$ and ω does therefore not belong to U. A contradiction. ■

Section two shows that long run multiple equilibria exist for some economies. Fuchs [8] shows that the number of equilibria is finite for the deterministic production model. This result generalizes to the production model $P(L)$ with uncertainty. This section shows that associated with every long run equilibrium of the production model $P(L)$ there exist "possibly" multiple short run equilibria of model $P(S)$ with the property that the number of short run equilibria is odd and finite. Oddness of equilibria follows from a straight forward application of degree theory along the lines of section three.

5. CONCLUSION

The paper shows that the application of the natural projection approach to the study of economic equilibrium is not restricted to pure exchange economies. It generalizes this approach to the study of economic equilibrium to the private ownership production model with time and uncertainty. Existence of equilibria of the production model is a consequence of the natural projection being smooth and proper. The structure of the set $\mathcal{E}$ is studied in some detail. It is shown that for a version of the Arrow-Debreu private ownership model with time and uncertainty the number of short run equilibria associated with every long run equilibrium is odd and finite. This model is particularly interesting since the generalized real asset structure allows for a richer interpretation of the model of the firm in terms of long and short run activities. In addition this model can easily be generalized beyond two periods. For that it suffices to consider a production set of the form $\bar{Y}_j = \big(y(0), y(1)\big) \in \mathbb{R}^a_- \times \mathbb{R}^b_+ \times \mathbb{R}^{aS}_- \times \mathbb{R}^{bS}_+ : F\big(y(0), y(1)\big) \leq 0$ with sign constraints on capital and production goods in every time period. For example, $y_j(t) = \left(y_j^1(t), \ldots, y_j^k(t), y_j^{k+1}(t), \ldots, y_j^l(t) \in \mathbb{R}_-^a \times \mathbb{R}_+^b\right)$ in every $t \in \{0,1\}$, where $y_j(t) \leq 0$ for index $1, \ldots, k$ and $y_j^k(t) \geq 0$ for index $k+1, \ldots, l$. In such a model a firm purchases capital and produces goods in every period. Here, the economy ends in period $t = 1$ is zero.

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